

# Contraction and optimality properties of adaptive Legendre-Galerkin methods: the 1-dimensional case

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## Abstract

As a first step towards a mathematically rigorous understanding of adaptive spectral/ $hp$  discretizations of elliptic boundary-value problems, we study the performance of adaptive Legendre-Galerkin methods in one space dimension. These methods offer unlimited approximation power only restricted by solution and data regularity. Our investigation is inspired by a similar study that we recently carried out for Fourier-Galerkin methods in a periodic box. We first consider an “ideal” algorithm, which we prove to be convergent at a fixed rate. Next we enhance its performance, consistently with the expected fast error decay of high-order methods, by activating a larger set of degrees of freedom at each iteration. We guarantee optimality (in the non-linear approximation sense) by incorporating a coarsening step. Optimality is measured in terms of certain sparsity classes of the Gevrey type, which describe a (sub-)exponential decay of the best approximation error.

**Keywords:** Spectral methods, adaptivity, convergence, optimal cardinality.

## 1 Introduction

The mathematical theory of adaptive algorithms for approximating the solution of multidimensional elliptic PDEs is rather recent. The first convergence results of adaptive finite element methods (AFEM) have been proved by Dörfler [12] and Morin, Nochetto, and Siebert [15]. On the other hand, the first convergence rates were derived by Cohen, Dahmen, and DeVore [7] for wavelets in any dimensions  $d$ , and for finite element methods (AFEM) by Binev, Dahmen, and DeVore [3] for  $d = 2$  and Stevenson [19] for any  $d$ . The most comprehensive results for AFEM are those of Cascón, Kreuzer, Nochetto, and Siebert [6] for any  $d$  and  $L^2$  data, and Cohen, DeVore, and Nochetto [8] for  $d = 2$  and  $H^{-1}$  data. The key result of this theory is that AFEM delivers a convergence rate compatible with that of the approximation classes where the solution and data

belong. The recent results in [8] reveal that it is the approximation class of the solution that really matters. However, in all the above cases (practical wavelets and FEM) the convergence rates are limited by the approximation power of the method, which is finite and related to the polynomial degree of the basis functions or the number of their vanishing moments, as well as the regularity of the solution and data. We refer to the surveys [16] by Nochetto, Siebert and Veerer for AFEM and [20] by Stevenson for adaptive wavelets.

A similar study for adaptive high-order methods (such as spectral element methods or  $hp$  finite element methods) has started only recently and very few results are available. The ultimate goal is to design algorithms which *optimally* choose between  $h$ -refinement and  $p$ -enrichment, and for which a rigorous mathematical proof of rate of convergence and optimality can be given. This is a formidable task which requires, among other things, the study of sparsity classes tailored to functions which are (locally) infinitely smooth. Some rigorous mathematical results on the convergence of  $hp$ -methods for PDEs have been established recently in [18, 13, 4]. Although the numerical implementations of adaptive  $hp$  methods has now reached high levels of efficiency (see, e.g., [17]), the theoretical study of optimality has never been addressed. A first step in this direction has been accomplished in [5], where the contraction and the optimal cardinality properties of adaptive Fourier-Galerkin methods in a periodic box in  $\mathbb{R}^d$  with dimension  $d \geq 1$  are presented together with the analysis of suitable nonlinear approximation classes (the classical one corresponding to algebraic decay of the Fourier coefficients and another one associated with exponential decay).

The present paper represents a second step towards the study of optimality for high-order methods. We extend the method and the results contained in [5] to a non-periodic setting in one dimension. This is the closest situation to the periodic case, since an orthonormal basis is readily available (the so called Babuška-Shen basis formed by the anti-derivatives of the Legendre polynomials); together with the associated dual basis, it allows one to represent the norm of a function or a functional as a  $\ell_2$ -type norm of the vector of its coefficients. In addition, the stiffness matrix for smooth coefficients of the differential operator exhibits a quasi-sparsity behavior, i.e., an exponential decay of its entries as one goes away from the diagonal.

In this paper we only consider the case of an exponential decay of the best approximation error of the solution of the PDE; indeed this is the most relevant situation which motivates the use of a spectral/ $p$  method. Our approach relies on a careful analysis of the relation between the sparsity class of a function and the sparsity class of its image through the differential operator. As already pointed out in the analysis of the Fourier method [5], the discrepancy between the sparsity classes of the residual and the exact solution suggests the introduction of a coarsening step that guarantees the optimality of the computed approximation at the end of each adaptive iteration.

The multi-dimensional situation, which poses additional difficulties, is currently under investigation.

## 2 Legendre and Babuška-Shen bases

Let  $I = (-1, 1)$  and  $L_k(x)$ ,  $k \geq 0$ , be the  $k$ -th Legendre orthogonal polynomial in  $I$ , which satisfies  $\deg L_k = k$ ,  $L_k(1) = 1$  and

$$\int_I L_k(x) L_m(x) dx = \frac{2}{2k+1} \delta_{km}, \quad m \geq 0.$$

Furthermore, we denote by

$$\phi_k(x) = \sqrt{k+1/2} L_k(x), \quad k \geq 0,$$

the elements of the orthonormal *Legendre basis* in  $L^2(I)$ , which satisfy

$$\int_I \phi_k(x) \phi_m(x) dx = \delta_{km}, \quad m \geq 0.$$

We denote by  $D = d/dx$  the first derivative operator. The natural modal basis in  $H_0^1(I)$  is the *Babuška-Shen basis* (BS basis), whose elements are defined as

$$\eta_k(x) = \sqrt{k-1/2} \int_x^1 L_{k-1}(s) ds = \frac{1}{\sqrt{4k-2}} (L_{k-2}(x) - L_k(x)) \quad k \geq 2; \quad (2.1)$$

they satisfies  $\deg \eta_k = k$  and

$$D\eta_k = -\phi_{k-1}. \quad (2.2)$$

Thus, the  $\eta_k$ 's satisfy

$$(\eta_k, \eta_m)_{H_0^1(I)} = \int_I D\eta_k(x) D\eta_m(x) dx = \delta_{km}, \quad k, m \geq 2, \quad (2.3)$$

i.e., they form an orthonormal basis for the  $H_0^1(I)$ -inner product.

Equivalently, the (semi-infinite) stiffness matrix  $\mathbf{S}_\eta$  of the Babuška-Shen basis with respect to this inner product is the identity matrix  $\mathbf{I}$ . On the other hand, one has

$$(\eta_k, \eta_m)_{L^2(I)} = \begin{cases} \frac{2}{(2k-3)(2k+1)} & \text{if } m = k, \\ -\frac{1}{(2k+1)\sqrt{(2k-1)(2k+3)}} & \text{if } m = k+2, \\ 0 & \text{elsewhere.} \end{cases} \quad \text{for } k \geq m, \quad (2.4)$$

which means that the mass matrix  $\mathbf{M}_\eta$  is pentadiagonal. (Since even and odd modes are mutually orthogonal, the mass matrix could be equivalently represented by a couple of tridiagonal matrices, each one collecting the inner products of all modes with equal parity). For every  $v \in H_0^1(I)$  we have

$$v(x) = \sum_{k=2}^{\infty} \hat{v}_k \eta_k(x)$$

with  $\hat{v}_k = \int_{-1}^1 Dv(x) D\eta_k(x) dx$ . In view of the results in the next sections, we observe that (2.2) yields

$$Dv = \sum_{k=2}^{\infty} \hat{v}_k D\eta_k = - \sum_{k=2}^{\infty} \hat{v}_k \phi_{k-1}; \quad (2.5)$$

comparing this expression with

$$Dv = \sum_{h=1}^{\infty} (Dv)_h^\wedge \phi_h$$

yields

$$(Dv)_h^\wedge = -\hat{v}_{h+1} \quad \forall h \geq 1. \quad (2.6)$$

From (2.3), there follows that the  $H_0^1(I)$ -norm can be expressed, according to the Parseval identity, as

$$\|v\|_{H_0^1(I)}^2 = \sum_{k=2}^{\infty} |\hat{v}_k|^2 = \mathbf{v}^T \mathbf{v} , \quad (2.7)$$

where the vector  $\mathbf{v} = (\hat{v}_k)$  collects the coefficients of  $v$ . The  $L^2(I)$ -norm of  $v$  is given by

$$\|v\|_{L^2(I)}^2 = \mathbf{v}^T \mathbf{M}_\eta \mathbf{v} . \quad (2.8)$$

Correspondingly, any element  $f \in H^{-1}(I)$  can be expanded in terms of the *dual Babuška-Shen basis*, whose elements  $\eta_k^*$ ,  $k \geq 2$ , are defined by the conditions

$$\langle \eta_k^*, v \rangle = \hat{v}_k \quad \forall v \in H_0^1(I) ;$$

precisely one has

$$f = \sum_{k=2}^{\infty} \hat{f}_k \eta_k^* , \quad \text{with } \hat{f}_k = \langle f, \eta_k \rangle ,$$

and its  $H^{-1}(I)$ -norm can be expressed, according to the Parseval identity, as

$$\|f\|_{H^{-1}(I)}^2 = \sum_{k=2}^{\infty} |\hat{f}_k|^2 . \quad (2.9)$$

Summarizing, we see that the one-dimensional Legendre case is similar, from the point of view of expansions and norm representations, to the Fourier case (see [5]).

Throughout the paper, we will use the notation  $\| \cdot \|$  to indicate both the  $H_0^1(I)$ -norm of a function  $v$ , or the  $H^{-1}(I)$ -norm of a linear form  $f$ ; the specific meaning will be clear from the context.

Moreover, given any finite index set  $\Lambda \subset \mathbb{N}_2 := \{k \in \mathbb{N} : k \geq 2\}$ , we define the subspace of  $V := H_0^1(I)$

$$V_\Lambda := \text{span} \{ \eta_k \mid k \in \Lambda \} ;$$

we set  $|\Lambda| = \text{card } \Lambda$ , so that  $\dim V_\Lambda = |\Lambda|$ . If  $g$  admits an expansion  $g = \sum_{k=2}^{\infty} \hat{g}_k \eta_k$  (converging in an appropriate norm), then we define its projection  $P_\Lambda g$  onto  $V_\Lambda$  by setting

$$P_\Lambda g = \sum_{k \in \Lambda} \hat{g}_k \eta_k .$$

### 3 The model problem and its Galerkin discretization

We now consider the elliptic problem

$$\begin{cases} Lu = -D \cdot (\nu Du) + \sigma u = f & \text{in } I , \\ u(-1) = u(1) = 0 , \end{cases} \quad (3.1)$$

where  $\nu$  and  $\sigma$  are sufficiently smooth real coefficients satisfying  $0 < \nu_* \leq \nu(x) \leq \nu^* < \infty$  and  $0 \leq \sigma(x) \leq \sigma^* < \infty$  in  $I$ ; let us set

$$\alpha_* = \nu_* \quad \text{and} \quad \alpha^* = \max(\nu^*, \sigma^*) .$$

We formulate this problem variationally as

$$u \in H_0^1(I) \quad : \quad a(u, v) = \langle f, v \rangle \quad \forall v \in H_0^1(I) , \quad (3.2)$$

where  $a(u, v) = \int_I \nu Du Dv + \int_I \sigma uv$ . We denote by  $\|v\| = \sqrt{a(v, v)}$  the energy norm of any  $v \in H_0^1(I)$ , which satisfies

$$\sqrt{\alpha_*} \|v\| \leq \|v\| \leq \sqrt{\alpha^*} \|v\| . \quad (3.3)$$

Given any finite set  $\Lambda \subset \mathbb{N}_2$ , the Galerkin approximation is defined as

$$u_\Lambda \in V_\Lambda \quad : \quad a(u_\Lambda, v_\Lambda) = \langle f, v_\Lambda \rangle \quad \forall v_\Lambda \in V_\Lambda . \quad (3.4)$$

For any  $w \in V_\Lambda$ , we define the residual

$$r(w) = f - Lw = \sum_k \hat{r}_k(w) \eta_k^* , \quad \text{where} \quad \hat{r}_k(w) = \langle f - Lw, \eta_k \rangle = \langle f, \eta_k \rangle - a(w, \eta_k) .$$

Then, the previous definition of  $u_\Lambda$  is equivalent to the condition

$$P_\Lambda r(u_\Lambda) = 0 , \quad \text{i.e.,} \quad \hat{r}_k(u_\Lambda) = 0 \quad \forall k \in \Lambda . \quad (3.5)$$

On the other hand, by the continuity and coercivity of the bilinear form  $a$ , one has

$$\frac{1}{\alpha^*} \|r(u_\Lambda)\| \leq \|u - u_\Lambda\| \leq \frac{1}{\alpha_*} \|r(u_\Lambda)\| , \quad (3.6)$$

or, equivalently,

$$\frac{1}{\sqrt{\alpha^*}} \|r(u_\Lambda)\| \leq \|u - u_\Lambda\| \leq \frac{1}{\sqrt{\alpha_*}} \|r(u_\Lambda)\| . \quad (3.7)$$

### 3.1 Algebraic representation and properties of the stiffness matrix

Let us identify the solution  $u = \sum_k \hat{u}_k \eta_k$  of Problem (3.2) with the vector  $\mathbf{u} = (\hat{u}_k)$  of its Babuška-Shen (BS) coefficients. Similarly, let us identify the right-hand side  $f$  with the vector  $\mathbf{f} = (\hat{f}_\ell)$  of its BS coefficients. Finally, let us introduce the semi-infinite, symmetric and positive-definite matrix

$$\mathbf{A} = (a_{\ell, k}) \quad \text{with} \quad a_{\ell, k} = a(\eta_k, \eta_\ell)_{k, \ell \geq 2} . \quad (3.8)$$

Then, Problem (3.2) can be equivalently written as

$$\mathbf{A} \mathbf{u} = \mathbf{f} . \quad (3.9)$$

In order to study the properties of the matrix  $\mathbf{A}$ , let us first recall the following result.

**Property 3.1** (product of Legendre polynomials). *There holds*

$$L_m(x) L_n(x) = \sum_{r=0}^{\min(m, n)} A_{m, n}^r L_{m+n-2r}(x) \quad (3.10)$$

with

$$A_{m, n}^r := \frac{A_{m-r} A_r A_{n-r}}{A_{n+m-r}} \frac{2n + 2m - 4r + 1}{2n + 2m - 2r + 1}$$

and

$$A_0 := 1 , \quad A_m := \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{m!} = \frac{(2m)!}{2^m (m!)^2} .$$

*Proof.* See e.g. [1]. □

Bearing in mind (3.8), in the sequel we will make use of the following notation

$$a_{m,n} = a_{m,n}^{(1)} + a_{m,n}^{(0)}, \quad m, n \geq 2$$

with

$$a_{m,n}^{(1)} := \int_{-1}^1 \nu(x) D\eta_m(x) D\eta_n(x) dx \quad \text{and} \quad a_{m,n}^{(0)} := \int_{-1}^1 \sigma(x) \eta_m(x) \eta_n(x) dx.$$

If  $\nu(x) = \sum_{k=0}^{\infty} \nu_k L_k$ , then using (3.10) it is possible to prove that

$$\begin{aligned} a_{m+1,n+1}^{(1)} &:= \int_{-1}^1 \nu(x) D\eta_{m+1}(x) D\eta_{n+1}(x) dx \\ &= \frac{\sqrt{(2m+1)(2n+1)}}{2} \int_{-1}^1 \nu(x) L_m(x) L_n(x) dx \\ &= \sum_{r=0}^{\min(m,n)} B_{m,n}^r \nu_{m+n-2r}, \end{aligned} \tag{3.11}$$

where we set

$$B_{m,n}^r := \frac{\sqrt{(2m+1)(2n+1)}}{2m+2n-4r+1} A_{m,n}^r. \tag{3.12}$$

**Property 3.2** (coefficients  $B_{m,n}^r$ ). *There holds*

$$B_{m,n}^r \lesssim 1$$

for every  $m, n \geq 0$  and  $0 \leq r \leq \min(m, n)$ .

*Proof.* We first observe that Stirling's formula  $m! \sim \sqrt{2\pi m} e^{-m} m^m$  implies

$$A_m \sim \frac{2^m}{\sqrt{\pi m}}. \tag{3.13}$$

This can be used to prove asymptotic estimates for the factor  $\frac{A_{m-r} A_r A_{n-r}}{A_{n+m-r}}$  of  $A_{m,n}^r$ :

- Case  $0 < r < \min(m, n)$ :

$$\frac{A_{m-r} A_r A_{n-r}}{A_{n+m-r}} \sim \frac{1}{\pi} \frac{\sqrt{n+m-r}}{\sqrt{m-r} \sqrt{n-r} \sqrt{r}};$$

- Case  $r = 0$ :

$$\frac{A_m A_n}{A_{n+m}} \sim \frac{1}{\sqrt{\pi}} \frac{\sqrt{n+m}}{\sqrt{nm}};$$

- Case  $r = \min(m, n)$  and  $m \neq n$ :

$$\frac{A_{\min(m,n)} A_{|m-n|}}{A_{\max(m,n)}} \sim \frac{1}{\sqrt{\pi}} \frac{\sqrt{\max(m,n)}}{\sqrt{\min(m,n)} \sqrt{|m-n|}}.$$

When  $m = n$  it is sufficient to use  $A_0 = 1$  to get  $\frac{A_m A_0}{A_m} = 1$ .

Combining these results with (3.12), we now estimate the quantities  $B_{m,n}^r$  as follows:

(a) Case  $0 < r < \min(m, n)$ :

$$\begin{aligned} B_{m,n}^r &\sim \frac{1}{\pi} \frac{\sqrt{(2m+1)(2n+1)}}{2m+2n-2r+1} \frac{\sqrt{n+m-r}}{\sqrt{(m-r)(n-r)r}} \\ &\sim \frac{1}{\pi} \frac{\sqrt{mn}}{\sqrt{m+n-r}} \frac{1}{\sqrt{m-r}\sqrt{n-r}\sqrt{r}} . \end{aligned}$$

We note that the above last term can be asymptotically bounded by

$$B_{m,n}^r \sim \frac{1}{\pi} \frac{1}{\sqrt{\min(m+n, |m-n|)}} ,$$

which amounts to considering the extremal cases  $r = 1$  and  $r = \min(m, n) - 1$ .

(b) Case  $r = 0$ :

$$B_{m,n}^0 \sim \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{m+n}} .$$

(c) Case  $r = \min(m, n)$  and  $m \neq n$ :

$$B_{m,n}^{\min(m,n)} \sim \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{|n-m|}} .$$

When  $m = n$  we obtain  $B_{m,m}^m \sim 1$ .

Hence, by combining (a)-(c), there follows that the terms  $B_{m,n}^r$ ,  $0 \leq r \leq \min(m, n)$  are asymptotically bounded by a constant independent of  $m, n, r$ .  $\square$

**Proposition 3.1** (decay of  $a_{m,n}^{(1)}$ ). *If there exists  $\eta > 0$  and a positive constant  $C_\eta$  only depending on  $\eta$  such that*

$$|\nu_k| \leq C_\eta e^{-\eta k} \quad \forall k \geq 0 ,$$

*then there holds*

$$|a_{m,n}^{(1)}| \leq C e^{-\eta|n-m|} \quad \forall n, m \geq 2 , \quad (3.14)$$

*where  $C$  is a constant only depending on  $\eta$ .*

*Proof.* Using (3.11) and Property 3.2 there follows

$$\begin{aligned} |a_{m,n}^{(1)}| &\lesssim \sum_{r=0}^{\min(m,n)} |\nu_{m+n-2r}| \lesssim \sum_{r=0}^{\min(m,n)} e^{-\eta(m+n-2r)} \\ &\lesssim e^{-\eta|m-n|} \sum_{r=0}^{\min(m,n)} e^{-2\eta(\min(m,n)-r)} \lesssim C_\eta e^{-\eta|m-n|} . \end{aligned}$$

This gives (3.14) as asserted.  $\square$

Let  $\sigma(x) = \sum_{k=0}^{\infty} \sigma_k L_k$  then using (2.1) and (3.10) it is possible to prove that

$$\begin{aligned}
a_{m,n}^{(0)} &:= \int_{-1}^1 \sigma(x) \eta_m(x) \eta_n(x) dx \\
&= \frac{1}{\sqrt{(2m-1)(2n-1)}} \int_{-1}^1 \sigma(x) (L_{m-2}(x) - L_m(x)) (L_{n-2}(x) - L_n(x)) dx \\
&= \frac{1}{\sqrt{(2m-1)(2n-1)}} \left\{ \sum_{r=0}^{\min(m-2, n-2)} C_{m-2, n-2}^r \sigma_{m-2+n-2-2r} \right. \\
&\quad + \sum_{r=0}^{\min(m-2, n)} C_{m-2, n}^r \sigma_{m-2+n-2r} + \sum_{r=0}^{\min(m, n-2)} C_{m, n-2}^r \sigma_{m+n-2-2r} \\
&\quad \left. + \sum_{r=0}^{\min(m, n)} C_{m, n}^r \sigma_{m+n-2r} \right\}, \tag{3.15}
\end{aligned}$$

where

$$C_{m,n}^r := \frac{A_{m,n}^r}{2m+2n-4r+1}.$$

**Property 3.3** (coefficients  $C_{m,n}^r$ ). *There holds*

$$\frac{1}{\sqrt{(2m-1)(2n-1)}} C_{m-k, n-j}^r \lesssim 1$$

with  $k, j = 0, 2$  and  $m, n \geq 0$  and  $0 \leq r \leq \min(m-k, n-j)$ .

*Proof.* Let us first consider the case  $k, j = 0$ . Proceeding as in the proof of Property 3.2 we get the following cases for the auxiliary quantity  $D_{m,n}^r = \frac{1}{\sqrt{(2m-1)(2n-1)}} C_{m,n}^r$ :

(a) Case  $0 < r < \min(m, n)$ :

$$D_{m,n}^r \sim \frac{1}{\pi} \frac{1}{\sqrt{(2m-1)(2n-1)}} \frac{1}{2m+2n-2r+1} \frac{\sqrt{n+m-r}}{\sqrt{(m-r)(n-r)r}}.$$

(b) Case  $r = 0$ :

$$D_{m,n}^r \sim \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{(2m-1)(2n-1)}} \frac{1}{2m+2n+1} \frac{\sqrt{n+m}}{\sqrt{mn}}.$$

(c) Case  $r = \min(m, n)$ :

$$D_{m,n}^r \sim \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{(2m-1)(2n-1)}} \frac{1}{2\max(m, n)+1} \frac{\sqrt{\max(n, m)}}{\sqrt{\min(m, n)}\sqrt{|m-n|}}.$$

We note that each right-hand side term in (a)-(c) is asymptotically bounded by a constant. Therefore the quantities  $D_{m,n}^r$ ,  $0 \leq r \leq \min(m, n)$  are asymptotically bounded by a constant independent of  $n, m, r$ . The other terms in (3.15) can be treated similarly.  $\square$



**Proposition 3.2** (decay of  $a_{m,n}^{(0)}$ ). *If there exists  $\eta > 0$  and a positive constant  $C_\eta$  only depending on  $\eta$  such that*

$$|\sigma_k| \leq C_\eta e^{-\eta k} \quad \forall k \geq 0 ,$$

*then there holds*

$$|a_{m,n}^{(0)}| \leq C e^{-\eta|n-m|} \quad \forall n, m \geq 0 , \quad (3.16)$$

*where  $C$  is a constant only depending on  $\eta$ .*

*Proof.* Use (3.15) together with Property 3.3 and argue as in the proof of Proposition 3.1.  $\square$

Combining Propositions 3.1 and 3.2 yields

**Corollary 3.1** (decay of  $a_{m,n}$ ). *If there exists  $\eta > 0$  and a positive constant  $C_\eta$  only depending on  $\eta$  such that*

$$|\nu_k|, |\sigma_k| \leq C_\eta e^{-\eta k} \quad \forall k \geq 0 ,$$

*then there holds*

$$|a_{m,n}| \leq C e^{-\eta|n-m|} \quad \forall n, m \geq 2 , \quad (3.17)$$

*where  $C$  is a constant only depending on  $\eta$ .*

Correspondingly, the matrix  $\mathbf{A}$  belongs to the following class.

**Definition 3.1** (regularity classes for  $\mathbf{A}$ ). *A matrix  $\mathbf{A}$  is said to belong to the exponential class  $\mathcal{D}_e(\eta_L)$  if there exists a constant  $c_L > 0$  such that its elements satisfy*

$$|a_{m,n}| \leq c_L e^{-\eta_L |m-n|} \quad m, n \geq 2 . \quad (3.18)$$

The following properties hold.

**Property 3.4** (continuity of  $\mathbf{A}$ ). *If  $\mathbf{A} \in \mathcal{D}_e(\eta_L)$ , then  $\mathbf{A}$  defines a bounded operator on  $\ell^2(\mathbb{N}_2)$ .*

*Proof.* It is sufficient to extend the semi-infinite matrix  $\mathbf{A} = (a_{\ell,k})_{\ell,k \in \mathbb{N}_2}$  to a bi-infinite matrix  $\tilde{\mathbf{A}} = (\tilde{a}_{\ell,k})_{\ell,k \in \mathbb{Z}}$  such that it corresponds to the identity matrix for  $\ell, k \in \mathbb{Z} \setminus \mathbb{N}_2$  and it is equal to  $\mathbf{A}$  otherwise. Then proceed as in [14, 10].  $\square$

**Property 3.5** (inverse of  $\mathbf{A}$ ). *If  $\mathbf{A} \in \mathcal{D}_e(\eta_L)$  and there exists a constant  $c_L$  satisfying (3.18) such that*

$$c_L < \frac{1}{2}(e^{\eta_L} - 1) \min_{\ell} a_{\ell,\ell} , \quad (3.19)$$

*then  $\mathbf{A}$  is invertible in  $\ell^2(\mathbb{N}_2)$  and  $\mathbf{A}^{-1} \in \mathcal{D}_e(\bar{\eta}_L)$  where  $\bar{\eta}_L \in (0, \eta_L]$  is such that  $\bar{z} = e^{-\bar{\eta}_L}$  is the unique zero in the interval  $(0, 1)$  of the polynomial*

$$z^2 - \frac{e^{2\eta_L} + 2c_L + 1}{e^{\eta_L}(c_L + 1)} z + 1 .$$

*Proof.* See [5, Property 2.3].  $\square$

For any integer  $J \geq 0$ , let  $\mathbf{A}_J$  denote the following symmetric truncation of the matrix  $\mathbf{A}$

$$(\mathbf{A}_J)_{\ell,k} = \begin{cases} a_{\ell,k} & \text{if } |\ell - k| \leq J , \\ 0 & \text{elsewhere.} \end{cases} \quad (3.20)$$

Then, we have the following well-known results.

**Property 3.6** (truncation). *The truncated matrix  $\mathbf{A}_J$  has a number of non-vanishing entries bounded by  $2J$ . Moreover, under the assumption of Property 3.4, there exists a constant  $C_{\mathbf{A}}$  such that*

$$\|\mathbf{A} - \mathbf{A}_J\| \leq \psi_{\mathbf{A}}(J, \eta) := C_{\mathbf{A}} e^{-\eta_L J}$$

for all  $J \geq 0$ . Consequently, under the assumptions of Property 3.5, one has

$$\|\mathbf{A}^{-1} - (\mathbf{A}^{-1})_J\| \leq \psi_{\mathbf{A}^{-1}}(J, \bar{\eta}_L) \quad (3.21)$$

where we let  $\bar{\eta}_L$  be defined in Property 3.5.

*Proof.* See [5, Property 2.4]. □

## 4 Towards an adaptive algorithm

In order to design an adaptive algorithm with optimal convergence and complexity properties, we start by considering an *ideal one*. This will serve as a reference to discuss the most relevant aspects which have to be taken into account in designing the final algorithm. The ideal algorithm uses as error estimator the ideal one, i.e., the norm of the residual in  $H^{-1}(I)$ . We thus set, for any  $v \in H_0^1(I)$ ,

$$\eta^2(v) = \|r(v)\|^2 = \sum_{k \in \mathbb{N}_2} |\hat{r}_k(v)|^2, \quad (4.1)$$

so that (3.6) can be rephrased as

$$\frac{1}{\alpha^*} \eta(u_{\Lambda}) \leq \|u - u_{\Lambda}\| \leq \frac{1}{\alpha_*} \eta(u_{\Lambda}). \quad (4.2)$$

Obviously, this estimator is hardly computable in practice. However, by introducing suitable polynomial approximations of the coefficients and the right-hand side, it is possible to consider a *feasible* version. In the sequel we will not pursue this possibility, but we refer to [5] for the details. Given any subset  $\Lambda \subseteq \mathbb{N}_2$ , we also define the quantity

$$\eta^2(v; \Lambda) = \|P_{\Lambda} r(v)\|^2 = \sum_{k \in \Lambda} |\hat{r}_k(v)|^2,$$

so that  $\eta(v) = \eta(v; \mathbb{N}_2)$ .

We now introduce the following procedures, which will enter the definition of all our adaptive algorithms.

- $u_{\Lambda} := \mathbf{GAL}(\Lambda)$   
Given a finite subset  $\Lambda \subset \mathbb{N}_2$ , the output  $u_{\Lambda} \in V_{\Lambda}$  is the solution of the Galerkin problem (3.4) relative to  $\Lambda$ .
- $r := \mathbf{RES}(v_{\Lambda})$   
Given a function  $v_{\Lambda} \in V_{\Lambda}$  for some finite index set  $\Lambda$ , the output  $r$  is the residual  $r(v_{\Lambda}) = f - Lv_{\Lambda}$ .

- $\Lambda^* := \mathbf{DÖRFLER}(r, \theta)$

Given  $\theta \in (0, 1)$  and an element  $r \in H^{-1}(I)$ , the output  $\Lambda^* \subset \mathbb{N}_2$  is a finite set of minimal cardinality such that the inequality

$$\|P_{\Lambda^*} r\| \geq \theta \|r\| \quad (4.3)$$

is satisfied.

Note that the latter inequality is equivalent to

$$\|r - P_{\Lambda^*} r\| \leq \sqrt{1 - \theta^2} \|r\|. \quad (4.4)$$

If  $r = r(u_\Lambda)$  is the residual of a Galerkin solution  $u_\Lambda \in V_\Lambda$ , then by (3.5) we can trivially assume that  $\Lambda^*$  is contained in  $\Lambda^c := \mathbb{N}_2 \setminus \Lambda$ . For such a residual, inequality (4.3) can then be stated as

$$\eta(u_\Lambda; \Lambda^*) \geq \theta \eta(u_\Lambda) \quad (4.5)$$

**Dörfler marking** (or bulk chasing). Writing  $\hat{r}_k = \hat{r}_k(u_\Lambda)$ , the condition (4.5) can be equivalently stated as

$$\sum_{k \in \Lambda^*} |\hat{r}_k|^2 \geq \theta^2 \sum_{k \notin \Lambda} |\hat{r}_k|^2. \quad (4.6)$$

Thus, the output set  $\Lambda^*$  of minimal cardinality can be immediately determined if the coefficients  $\hat{r}_k$  are rearranged in non-increasing order of modulus. The cardinality of  $\Lambda^*$  depends on the rate of decay of the rearranged coefficients, i.e., on the *sparsity* of the representation of the residual in the chosen basis.

Given two parameters  $\theta \in (0, 1)$  and  $tol \in [0, 1)$ , we are ready to define our ideal adaptive algorithm.

**Algorithm ADLEG**( $\theta, tol$ )

Set  $r_0 := f$ ,  $\Lambda_0 := \emptyset$ ,  $n = -1$

do

$n \leftarrow n + 1$

$\partial\Lambda_n := \mathbf{DÖRFLER}(r_n, \theta)$

$\Lambda_{n+1} := \Lambda_n \cup \partial\Lambda_n$

$u_{n+1} := \mathbf{GAL}(\Lambda_{n+1})$

$r_{n+1} := \mathbf{RES}(u_{n+1})$

while  $\|r_{n+1}\| > tol$

This algorithm is the non-periodic counterpart of the ideal algorithm **ADFOUR** considered in [5]. The same proof given therein yields the following result, which states the convergence of the algorithm with a guaranteed error reduction rate.

**Theorem 4.1** (contraction property of **ADLEG**). *Let us set*

$$\rho = \rho(\theta) = \sqrt{1 - \frac{\alpha_*}{\alpha^*} \theta^2} \in (0, 1) . \quad (4.7)$$

*Let  $\{\Lambda_n, u_n\}_{n \geq 0}$  be the sequence generated by the adaptive algorithm **ADLEG**. Then, the following bound holds for any  $n$ :*

$$\|u - u_{n+1}\| \leq \rho \|u - u_n\| .$$

*Thus, for any  $\text{tol} > 0$  the algorithm terminates in a finite number of iterations, whereas for  $\text{tol} = 0$  the sequence  $u_n$  converges to  $u$  in  $H^1(I)$  as  $n \rightarrow \infty$ .  $\square$*

At this point, some comments are in order.

- The predicted error reduction rate  $\rho = \rho(\theta)$ , being bounded from below by the quantity  $\sqrt{1 - \frac{\alpha_*}{\alpha^*}}$ , looks overly pessimistic, particularly in the case of smooth (analytic) solutions. Indeed, in this case a spectral (Legendre) Galerkin method is expected to exhibit very fast (exponential) error decay. For this reason, we will introduce in Section 7.3 a variant of the Dörfler procedure which – through a suitable enrichment of the set of new degrees of freedom detected by the usual Dörfler – will guarantee an arbitrarily large error reduction per iteration.
- The complexity analysis of the algorithm requires to relate the current error  $\varepsilon_n := \|u - u_n\|$  to the cardinality of the set  $\Lambda_n$  of the activated degrees of freedom, having as a benchmark the best approximation (i.e. the one achieved with the minimal number of degrees of freedom) of the exact solution  $u$  up to an error given exactly by  $\varepsilon_n$ . This requires to investigate the *sparsity class* of the solution  $u$ , a task that will be accomplished in Section 5; we will confine ourselves to the case of infinite differentiable functions (including analytic ones) for which a natural framework is provided by Gevrey spaces. (The analysis of the case of finite smoothness can be carried out by extending the arguments presented in [5].)
- The cardinality of the set  $\Lambda_n$  of active degrees of freedom selected by the Dörfler procedure may be estimated in terms of the sparsity class of the residual, rather than the one of the solution. If the residual is less sparse than the solution, we run into a potential situation of non-optimality. This is precisely what happens for the Gevrey-type sparsity classes, as pointed out in Section 6. For this reason, we will incorporate in our algorithm a coarsening step, introduced in Section 7.2, to bring the cardinality of the active degrees of freedom at the end of each iteration to be comparable with the optimal one dictated by the sparsity class of the solution.

## 5 Nonlinear approximation in Gevrey spaces

At first, we consider Gevrey spaces of linear type and then, through the concept of nonlinear approximation, we will introduce sparsity classes of functions related to Gevrey spaces.

### 5.1 Gevrey classes and their properties

We recall the following definition of classical Gevrey space.

**Definition 5.1** (definition of  $G^t(\bar{I})$ ). For any  $t \in (0, 1]$ , we denote by  $G^t(\bar{I})$  the space of  $C^\infty$  functions  $v$  in a neighborhood of  $\bar{I}$  for which there exist a constant  $L \geq 0$  such that for any  $n \geq 0$

$$\|D^n v\|_{L^2(I)} \leq L^{n+1}(n!)^{1/t} . \quad (5.1)$$

The choice  $t = 1$  yields the usual class  $\mathcal{A}(\bar{I}) = G^1(\bar{I})$  of analytic functions in a neighborhood of  $\bar{I}$ .

Another family of spaces of Gevrey type has been introduced in [2] by relaxing the requirement on the growth of the derivatives near the boundary. In our simple one-dimensional setting, its definition is as follows.

**Definition 5.2** (definition of  $\mathcal{A}^t(\bar{I})$ ). Let  $\mathcal{L}$  denote the Legendre operator in  $I$

$$\mathcal{L}v = -D((1-x^2)D)v ;$$

and, for any  $k \geq 0$ , let us set

$$R_k v = \begin{cases} \mathcal{L}^p v & \text{if } k = 2p, \\ (1-x^2)D\mathcal{L}^p v & \text{if } k = 2p+1 . \end{cases}$$

Then for any  $t \in (0, 1]$ , we denote by  $\mathcal{A}^t(\bar{I})$  the space of  $C^\infty$  functions  $v$  in a neighborhood of  $\bar{I}$  for which there exist a constant  $L \geq 0$  such that for any  $k \geq 0$

$$\|R_k v\|_{L^2(I)} \leq L^{k+1}(k!)^{1/t} . \quad (5.2)$$

The relation between the two families is given by the inclusion

$$G^t(\bar{I}) \subseteq \mathcal{A}^t(\bar{I}) \subseteq G^{\frac{t}{2-t}}(\bar{I}) , \quad (5.3)$$

where the first inclusion is an immediate consequence of the fact that  $1-x^2$  is bounded with its derivatives in  $\bar{I}$ , whereas the second one is proved in [2].

Let  $\mathbb{P}_n = \{\phi_0, \dots, \phi_n\}$  be the space of polynomials of degree  $\leq n$  in  $I$  and let  $d_2(v, \mathbb{P}_n) = \inf_{p \in \mathbb{P}_n} \|v - p\|_{L^2(I)}$  be the distance of a function  $v \in L^2(I)$  to  $\mathbb{P}_n$ .

**Proposition 5.1.** Let  $v \in L^2(I)$ . The following conditions are equivalent:

- (i)  $v \in \mathcal{A}^t(I)$ .
- (ii) There exist  $L_1 \geq 0$  and  $\eta_1 > 0$  such that

$$d_2(v, \mathbb{P}_n) \leq L_1 e^{-\eta_1 n^t} \quad \forall n \geq 0. \quad (5.4)$$

- (iii) There exist  $L_2 \geq 0$  and  $\eta_2 > 0$  such that the Legendre coefficients of  $v = \sum_{k=0}^{\infty} \hat{v}_k \phi_k$  satisfy

$$|\hat{v}_k| \leq L_2 e^{-\eta_2 k^t} \quad \forall k \geq 0. \quad (5.5)$$

*Proof.* The equivalence (i)  $\leftrightarrow$  (ii) is proved in [2, Thm. 7.2]. In order to prove the equivalence (ii)  $\leftrightarrow$  (iii), we observe that

$$d_2(v, \mathbb{P}_n) = \|v - \sum_{k=0}^n \hat{v}_k \phi_k\|_{L^2(I)} = \left( \sum_{k>n} |\hat{v}_k|^2 \right)^{1/2} . \quad (5.6)$$

Thus, if (ii) holds, we have

$$|\hat{v}_{n+1}| \leq d_2(v, \mathbb{P}_n) \leq L_1 e^{-\eta_1 n^t}. \quad (5.7)$$

Now, recalling that  $(a+b)^t \leq a^t + b^t$  for  $a, b \geq 0$  and  $t \in (0, 1]$ , we have

$$e^{-\eta_1 n^t} = e^{\eta_1} e^{-\eta_1(n^t+1)} = e^{\eta_1} e^{-\eta_1(n^t+1^t)} \leq e^{\eta_1} e^{-\eta_1(n+1)^t} \quad (5.8)$$

which yields (iii) with  $L_2 = e^{\eta_1} L_1$  and  $\eta_2 = \eta_1$ . Conversely, let us assume that (iii) holds. Then, using again (5.4) and setting for simplicity  $\eta = \eta_2$ , we have

$$d_2(v, \mathbb{P}_n)^2 \leq L_2^2 \sum_{k>n} e^{-2\eta k^t} \lesssim L_2^2 \int_n^{+\infty} e^{-2\eta y^t} dy. \quad (5.9)$$

Now, setting  $z = y^t$  and  $s = 1/t \geq 1$ , we have

$$\begin{aligned} S_n &:= \int_n^{+\infty} e^{-2\eta y^t} dy = s \int_{n^t}^{+\infty} e^{-2\eta z} z^{s-1} dz \\ &= s e^{-2\eta n^t} \int_{n^t}^{+\infty} e^{-2\eta(z-n^t)} z^{s-1} dz = s e^{-2\eta n^t} \int_0^\infty e^{-2\eta w} (z+n^t)^{s-1} dw. \end{aligned} \quad (5.10)$$

The last integral can be bounded by a polynomial of degree  $[s-1]$ . It follows that for any  $\eta_1 < \eta = \eta_2$  we can find  $C_{\eta_1, s} > 0$  such that

$$S_n \leq C_{\eta_1, s} e^{-2\eta_1 n^t},$$

thus condition (ii) is satisfied.  $\square$

**Remark 5.1.** As a consequence of the previous proposition and the inclusion  $G^t(\bar{I}) \subseteq \mathcal{A}^t(\bar{I})$ , any Gevrey function  $v \in G^t(\bar{I})$  admits Legendre coefficients  $\hat{v}_k$  which decay according to the law (5.5).

For analytic functions, the results in (ii) and (iii) of Proposition 5.1 can be made more precise as follows.

**Proposition 5.2.** *Let  $v \in L^2(I)$  be analytic in the closed ellipse  $\mathcal{E}_r \supset [-1, 1]$  in the complex plane  $\mathbb{C}$  with foci in  $\pm 1$  and semiaxes' sum equal to  $r > 1$ . Then, one has*

$$|\hat{v}_k| \leq C(r) \sqrt{2(2k+1)} e^{-\eta k} \quad \forall k \geq 0, \quad (5.11)$$

and

$$d_2(v, \mathbb{P}_n) \leq \tilde{C}(r)(n+1)^{1/2} e^{-\eta n} \quad \forall n \geq 0 \quad (5.12)$$

for constants  $C(r)$  and  $\tilde{C}(r)$  only depending on  $r$  and  $\eta = \log r$ .

*Proof.* The estimate (5.11) is a consequence of the bound

$$|\hat{v}_k^L| \leq C(r)(2k+1) e^{-\eta k} \quad k \geq 0$$

given in [11, Theorem 12.4.7], where  $\hat{v}_k^L = (k+1/2) \int_{-1}^1 v(x) L_k(x) dx$  and satisfies  $\hat{v}_k^L = \sqrt{k+1/2} \hat{v}_k$ . On the other hand,

$$\begin{aligned} d_2(v, \mathbb{P}_n)^2 &= \sum_{k>n} |\hat{v}_k|^2 \leq 2C^2(r) \sum_{k>n} (2k+1) e^{-2\eta k} \\ &\leq 2C^2(r) \int_n^{+\infty} (2y+3) e^{-2\eta y} dy \leq 6 \frac{C^2(r)}{\eta} (n+1) e^{-2\eta n}, \end{aligned}$$

and (5.12) follows.  $\square$

There follows that conditions (ii) and (iii) are fulfilled with  $t = 1$  and any  $\eta_1, \eta_2 < \log r$ .

## 5.2 Nonlinear approximation and sparsity classes

From now on, we represent any  $v \in V$  according to the BS basis  $\{\eta_k\}_{k=2}^\infty$  as  $v = \sum_{k=2}^\infty \hat{v}_k \eta_k(x)$ . We recall that given any nonempty finite index set  $\Lambda \subset \mathbb{N}_2$  and the corresponding subspace  $V_\Lambda \subset V$  of dimension  $|\Lambda| = \text{card } \Lambda$ , the best approximation of  $v$  in  $V_\Lambda$  is the orthogonal projection of  $v$  upon  $V_\Lambda$ , i.e. the function  $P_\Lambda v = \sum_{k \in \Lambda} \hat{v}_k \eta_k$ , which satisfies

$$\|v - P_\Lambda v\| = \left( \sum_{k \notin \Lambda} |\hat{v}_k|^2 \right)^{1/2}$$

(we set  $P_\Lambda v = 0$  if  $\Lambda = \emptyset$ ). For any integer  $N \geq 1$ , we minimize this error over all possible choices of  $\Lambda$  with cardinality  $N$ , thereby leading to the *best  $N$ -term approximation error*

$$E_N(v) = \inf_{\Lambda \subset \mathbb{N}_2, |\Lambda|=N} \|v - P_\Lambda v\|.$$

A way to construct a *best  $N$ -term approximation*  $v_N$  of  $v$  consists of rearranging the coefficients of  $v$  in decreasing order of modulus

$$|\hat{v}_{k_1}| \geq \dots \geq |\hat{v}_{k_n}| \geq |\hat{v}_{k_{n+1}}| \geq \dots$$

and setting  $v_N = P_{\Lambda_N} v$  with  $\Lambda_N = \{k_n : 1 \leq n \leq N\}$ . Let us denote from now on  $v_n^* = \hat{v}_{k_n}$  the rearranged BS coefficients of  $v$ . Then,

$$E_N(v) = \|v - P_{\Lambda_N} v\| = \left( \sum_{n > N} |v_n^*|^2 \right)^{1/2}.$$

We will be interested in classifying functions according to the decay law of their best  $N$ -term approximations, as  $N \rightarrow \infty$ . To this end, given a strictly decreasing function  $\phi : \mathbb{N} \rightarrow \mathbb{R}_+$  such that  $\phi(0) = \phi_0$  for some  $\phi_0 > 0$  and  $\phi(N) \rightarrow 0$  when  $N \rightarrow \infty$ , we introduce the corresponding *sparsity class*  $\mathcal{A}_\phi$  by setting

$$\mathcal{A}_\phi = \left\{ v \in V : \|v\|_{\mathcal{A}_\phi} := \sup_{N \geq 0} \frac{E_N(v)}{\phi(N)} < +\infty \right\}. \quad (5.13)$$

The quantity  $\|v\|_{\mathcal{A}_\phi}$  (which need not be a (quasi-)norm, since  $\mathcal{A}_\phi$  need not be a linear space) dictates the minimal number  $N_\varepsilon$  of basis functions needed to approximate  $v$  with accuracy  $\varepsilon$ . In fact, from the relations

$$E_{N_\varepsilon}(v) \leq \varepsilon < E_{N_\varepsilon-1}(v) \leq \phi(N_\varepsilon - 1) \|v\|_{\mathcal{A}_\phi},$$

and the monotonicity of  $\phi$ , we obtain

$$N_\varepsilon \leq \phi^{-1} \left( \frac{\varepsilon}{\|v\|_{\mathcal{A}_\phi}} \right) + 1. \quad (5.14)$$

Hereafter, we focus on sparsity classes identified by exponentially decaying  $\phi$  of the type  $\phi(N) = e^{-\eta N^t}$  for real numbers  $\eta > 0$  and  $0 < t \leq 1$ . The following argument clarifies their

relationship with certain families of Gevrey spaces. Let us introduce the following spaces of Gevrey type: given real numbers  $\eta > 0$  and  $0 < t \leq 1$ , we set

$$G^{\eta,t}(I) = \{v \in V : \|v\|_{G,\eta,t} := \sum_{k=2}^{\infty} e^{2\eta k^t} |\hat{v}_k|^2 < +\infty\} . \quad (5.15)$$

We immediately observe that  $v \in G^{\eta,t}(I)$  implies

$$|\hat{v}_k|^2 \leq e^{-2\eta k^t} \|v\|_{G,\eta,t}^2 \quad \forall k \geq 2.$$

Using (2.6), we deduce from Proposition 5.1 that the first derivative of a function  $v \in G^{\eta,t}(I)$  belongs to the space  $\mathcal{A}^t(\bar{I})$  defined in Definition 5.2; in particular, by using (5.3),  $v'$  is a Gevrey function in  $G^{\frac{t}{2-t}}(\bar{I})$ .

Functions in  $G^{\eta,t}(I)$  can be approximated by the linear orthogonal projection on  $\mathbb{P}_M^0(I) := \mathbb{P}_M(I) \cap H_0^1(I)$

$$P_M v = \sum_{k=2}^M \hat{v}_k \eta_k ,$$

for which we have

$$\begin{aligned} \|v - P_M v\|^2 &= \sum_{k>M} |\hat{v}_k|^2 = \sum_{k>M} e^{-2\eta k^t} e^{2\eta k^t} |\hat{v}_k|^2 \\ &\leq e^{-2\eta M^t} \sum_{k>M} e^{2\eta k^t} |\hat{v}_k|^2 \leq e^{-2\eta M^t} \|v\|_{G,\eta,t}^2 . \end{aligned}$$

Setting  $N := \dim(\mathbb{P}_M^0) = M - 1$ , we have

$$E_N(v) \leq \|v - P_{N+1} v\| \lesssim e^{-\eta(N+1)^t} \|v\|_{G,\eta,t} \leq e^{-\eta N^t} \|v\|_{G,\eta,t} . \quad (5.16)$$

Hence, we are led to introduce the function

$$\phi(N) = e^{-\eta N^t} \quad (N \geq 1) , \quad (5.17)$$

whose inverse is given by

$$\phi^{-1}(\lambda) = \frac{1}{\eta^{1/t}} \left( \log \frac{1}{\lambda} \right)^{1/t} \quad (\lambda \leq 1) , \quad (5.18)$$

and to consider the corresponding class  $\mathcal{A}_\phi$  defined in (5.13), which contains  $G^{\eta,t}(I)$ .

**Definition 5.3** (exponential class of functions). *We denote by  $\mathcal{A}_G^{\eta,t}$  the set defined as*

$$\mathcal{A}_G^{\eta,t} := \left\{ v \in V : \|v\|_{\mathcal{A}_G^{\eta,t}} := \sup_{N \geq 0} E_N(v) e^{\eta N^t} < +\infty \right\} .$$

The class  $\mathcal{A}_G^{\eta,t}$  can be equivalently characterized in terms of behavior of rearranged sequences of BS coefficients.

**Definition 5.4** (exponential class of sequences). *Let  $\ell_G^{\eta,t}(\mathbb{N}_2)$  be the subset of sequences  $\mathbf{v} \in \ell^2(\mathbb{N}_2)$  so that*

$$\|\mathbf{v}\|_{\ell_G^{\eta,t}(\mathbb{N}_2)} := \sup_{n \geq 1} \left( n^{(1-t)/2} \exp(\eta n^t) |v_n^*| \right) < +\infty ,$$

where  $\mathbf{v}^* = (v_n^*)_{n=1}^\infty$  is the non-increasing rearrangement of  $\mathbf{v}$ .



The relationship between  $\mathcal{A}_G^{\eta,t}$  and  $\ell_G^{\eta,t}(\mathbb{N}_2)$  is stated in the following Proposition, whose proof is a straightforward adaptation of the one given in [5, Proposition 4.2].

**Proposition 5.3** (equivalence of exponential classes). *Given a function  $v \in V$  and the sequence  $\mathbf{v} = (\hat{v}_k)_{k \in \mathbb{N}_2}$  of its BS coefficients, one has  $v \in \mathcal{A}_G^{\eta,t}$  if and only if  $\mathbf{v} \in \ell_G^{\eta,t}(\mathbb{N}_2)$ , with*

$$\|v\|_{\mathcal{A}_G^{\eta,t}} \lesssim \|\mathbf{v}\|_{\ell_G^{\eta,t}(\mathbb{N}_2)} \lesssim \|v\|_{\mathcal{A}_G^{\eta,t}}.$$

As shown in [5] the set  $\ell_G^{\eta,t}(\mathbb{N}_2)$  is not a vector space, since it may happen that  $\mathbf{u}, \mathbf{v}$  belong to this set, whereas  $\mathbf{u} + \mathbf{v}$  does not. On the other hand, we have the following property (see [5, Lemma 4.1]).

**Property 5.1** (quasi-triangle inequality). *If  $\mathbf{u}_i \in \ell_G^{\eta_i,t}(\mathbb{N}_2)$  for  $i = 1, 2$ , then  $\mathbf{u}_1 + \mathbf{u}_2 \in \ell_G^{\eta,t}(\mathbb{N}_2)$  with*

$$\|\mathbf{u}_1 + \mathbf{u}_2\|_{\ell_G^{\eta,t}(\mathbb{N}_2)} \leq \|\mathbf{u}_1\|_{\ell_G^{\eta_1,t}(\mathbb{N}_2)} + \|\mathbf{u}_2\|_{\ell_G^{\eta_2,t}(\mathbb{N}_2)}, \quad \eta^{-\frac{1}{t}} = \eta_1^{-\frac{1}{t}} + \eta_2^{-\frac{1}{t}}.$$

## 6 Sparsity classes of the residual and the solution

This section is devoted to the study of the sparsity class of the residual  $r = r(u_\Lambda)$  produced by any Galerkin solution  $u_\Lambda \in V_\Lambda$ , and its connection with the sparsity class of the exact solution  $u$ . Indeed, the step

$$\partial\Lambda := \mathbf{DÖRFLER}(r, \theta)$$

selects a set  $\partial\Lambda$  of minimal cardinality in  $\Lambda^c$  for which  $\|r - P_{\partial\Lambda} r\| \leq \sqrt{1 - \theta^2} \|r\|$ . Thus, if  $r$  belongs to a certain sparsity class  $\mathcal{A}_{\bar{\phi}}$ , identified by a function  $\bar{\phi}$  according to (5.13), then (5.14) yields

$$|\partial\Lambda| \leq \bar{\phi}^{-1} \left( \frac{\sqrt{1 - \theta^2} \|r\|}{\|r\|_{\mathcal{A}_{\bar{\phi}}}} \right) + 1. \quad (6.1)$$

Specifically, if  $r \in \mathcal{A}_G^{\bar{\eta}, \bar{t}}$  for some  $\bar{\eta} > 0$  and  $\bar{t} > 0$ , we have by (5.18)

$$|\partial\Lambda| \leq \frac{1}{\bar{\eta}^{1/\bar{t}}} \left( \log \frac{\|r\|_{\mathcal{A}_G^{\bar{\eta}, \bar{t}}}}{\sqrt{1 - \theta^2} \|r\|} \right)^{1/\bar{t}} + 1. \quad (6.2)$$

We begin by investigating the sparsity class of the image  $Lv$ , when the function  $v$  belongs to the sparsity class  $\mathcal{A}_G^{\eta,t}$ . It turns out that the sparsity classes of  $v$  and  $Lv$  are equivalent, in view of Proposition 5.3, to the sparsity classes of the related vectors  $\mathbf{v}$  and  $\mathbf{A}\mathbf{v}$ , where  $\mathbf{A}$  is the stiffness matrix (3.8).

**Proposition 6.1** (continuity of  $L$  in  $\mathcal{A}_G^{\eta,t}$ ). *Let the differential operator  $L$  be such that the corresponding stiffness matrix satisfies  $\mathbf{A} \in \mathcal{D}_e(\eta_L)$  for some constant  $\eta_L > 0$ . Assume that  $v \in \mathcal{A}_G^{\eta,t}$  for some  $\eta > 0$  and  $t \in (0, 1]$ . Let one of the two following set of conditions be satisfied.*

(a) *If the matrix  $\mathbf{A}$  is banded with  $2p + 1$  non-zero diagonals, let us set*

$$\bar{\eta} = \frac{\eta}{(2p + 1)^t}, \quad \bar{t} = t.$$

(b) If the matrix  $\mathbf{A}$  is dense, but the coefficients  $\eta_L$  and  $\eta$  satisfy the inequality  $\eta < \eta_L$ , let us set

$$\bar{\eta} = \zeta(t)\eta, \quad \bar{t} = \frac{t}{1+t},$$

where we define

$$\zeta(t) := \left(\frac{1+t}{2}\right)^{\frac{t}{1+t}} \quad \forall 0 < t \leq 1. \quad (6.3)$$

Then, one has  $Lv \in \mathcal{A}_G^{\bar{\eta}, \bar{t}}$ , with

$$\|Lv\|_{\mathcal{A}_G^{\bar{\eta}, \bar{t}}} \lesssim \|v\|_{\mathcal{A}_G^{\eta, t}}. \quad (6.4)$$

*Proof.* The proof is an adaptation of a similar result in the periodic case given in [5, Proposition 5.3]; we report here the details for completeness. Let  $\mathbf{A}_J$  be the truncation of the stiffness matrix defined in (3.20); thus, by Property 3.6 we have  $\|\mathbf{A} - \mathbf{A}_J\| \leq C_{\mathbf{A}}e^{-\eta_L J}$ ,  $J \geq 0$ . On the other hand, for any  $j \geq 1$ , let  $\mathbf{v}_j = \mathbf{P}_j(\mathbf{v})$  be a best  $j$ -term approximation of  $\mathbf{v}$  (with  $\mathbf{v}_0 = 0$ ), which therefore satisfies  $\|\mathbf{v} - \mathbf{v}_j\| \leq e^{-\eta j^t} \|\mathbf{v}\|_{\ell_G^{\eta, t}(\mathbb{N}_2)}$ . Note that the difference  $\mathbf{v}_j - \mathbf{v}_{j-1}$  consists of a single component and satisfies as well

$$\|\mathbf{v}_j - \mathbf{v}_{j-1}\| \lesssim e^{-\eta j^t} \|\mathbf{v}\|_{\ell_G^{\eta, t}(\mathbb{N}_2)}.$$

Finally, let us introduce the function  $\chi : \mathbb{N} \rightarrow \mathbb{N}$  defined as  $\chi(j) = \lceil j^t \rceil$ , the smallest integer larger than or equal to  $j^t$ .

For any  $J \geq 1$ , let  $\mathbf{w}_J$  be the approximation of  $\mathbf{A}\mathbf{v}$  defined as

$$\mathbf{w}_J = \sum_{j=1}^J \mathbf{A}_{\chi(J-j)}(\mathbf{v}_j - \mathbf{v}_{j-1}).$$

Writing  $\mathbf{v} = \mathbf{v} - \mathbf{v}_J + \sum_{j=1}^J(\mathbf{v}_j - \mathbf{v}_{j-1})$ , we obtain

$$\mathbf{A}\mathbf{v} - \mathbf{w}_J = \mathbf{A}(\mathbf{v} - \mathbf{v}_J) + \sum_{j=1}^J (\mathbf{A} - \mathbf{A}_{\chi(J-j)})(\mathbf{v}_j - \mathbf{v}_{j-1}).$$

We now assume to be in Case (b). Since  $\mathbf{A} : \ell^2(\mathbb{N}_2) \rightarrow \ell^2(\mathbb{N}_2)$  is continuous, the last equation yields

$$\|\mathbf{A}\mathbf{v} - \mathbf{w}_J\| \lesssim \left( e^{-\eta J^t} + \sum_{j=1}^J e^{-(\eta_L \lceil (J-j)^t \rceil + \eta j^t)} \right) \|\mathbf{v}\|_{\ell_G^{\eta, t}(\mathbb{N}_2)}. \quad (6.5)$$

The exponents of the addends can be bounded from below as follows because  $t \leq 1$

$$\begin{aligned} \eta_L \lceil (J-j)^t \rceil + \eta j^t &= \eta_L \lceil (J-j)^t \rceil - \eta(J-j)^t + \eta((J-j)^t + j^t) \\ &\geq \eta_L(J-j)^t - \eta(J-j)^t + \eta((J-j) + j)^t \\ &= \beta(J-j)^t + \eta J^t, \end{aligned}$$

with  $\beta = \eta_L - \eta > 0$  by assumption. Then, (6.5) yields

$$\|\mathbf{A}\mathbf{v} - \mathbf{w}_J\| \lesssim \left( 1 + \sum_{j=0}^{J-1} e^{-\beta j^t} \right) e^{-\eta J^t} \|\mathbf{v}\|_{\ell_G^{\eta, t}(\mathbb{N}_2)} \lesssim e^{-\eta J^t} \|\mathbf{v}\|_{\ell_G^{\eta, t}(\mathbb{N}_2)}. \quad (6.6)$$

On the other hand, by construction  $\mathbf{w}_J$  belongs to a finite dimensional space  $\mathbf{V}_{\Lambda_J}$ , where

$$|\Lambda_J| \leq 2 \sum_{j=1}^J \chi(J-j) = 2 \sum_{j=0}^{J-1} \lceil j^t \rceil \sim \frac{2}{1+t} J^{1+t} \quad \text{as } J \rightarrow \infty. \quad (6.7)$$

This implies

$$\|\mathbf{A}\mathbf{v} - \mathbf{w}_J\| \lesssim e^{-\bar{\eta}|\Lambda_J|^{\bar{t}}} \|\mathbf{v}\|_{\ell_G^{\eta,t}(\mathbb{N}_2)},$$

with  $\bar{t} = \frac{t}{1+t}$  and  $\bar{\eta} = \left(\frac{1+t}{2}\right)^{\bar{t}} \eta = \zeta(t)\eta$  as asserted.

We last consider Case (a). Since  $\mathbf{A}_{\chi(J-j)} = \mathbf{A}$  if  $\chi(J-j) \geq p$ , for  $\chi(J-j) \leq p$ , whence  $j \geq J - p^{1/t}$ , the summation in (6.5) can be limited to those  $j$  satisfying  $j_p \leq j \leq J$ , where  $j_p = \lceil J - p^{1/t} \rceil$ . Therefore

$$\|\mathbf{A}\mathbf{v} - \mathbf{w}_J\| \lesssim \left( e^{-\eta J^t} + \sum_{j=j_p}^J e^{-\eta j^t} \right) \|\mathbf{v}\|_{\ell_G^{\eta,t}(\mathbb{N}_2)}.$$

Now,  $J - j \leq p^{1/t}$  if  $j_p \leq j \leq J$  and  $j^t \geq j_p^t \geq (J - p^{1/t})^t \geq J^t - p$ , whence

$$\|\mathbf{A}\mathbf{v} - \mathbf{w}_J\| \lesssim (1 + e^{\eta p}) e^{-\eta J^t} \|\mathbf{v}\|_{\ell_G^{\eta,t}(\mathbb{N}_2)}.$$

We conclude by observing that  $|\Lambda_J| \leq (2p+1)J$ , since any matrix  $\mathbf{A}_J$  has at most  $2p+1$  diagonals.  $\square$

Before studying the sparsity class of the residual, it is convenient to rewrite the Galerkin problem (3.4) in an equivalent (infinite-dimensional) way. To this end, let  $\mathbf{u}_\Lambda \in \mathbb{R}^{|\Lambda|}$  be the vector collecting the coefficients of  $u_\Lambda$  indexed in  $\Lambda$ ; let  $\mathbf{f}_\Lambda \in \mathbb{R}^{|\Lambda|}$  be the analogous restriction for the vector of the coefficients of  $f$ . Finally, denote by  $\mathbf{R}_\Lambda$  the matrix that restricts a semi-infinite vector to the portion indexed in  $\Lambda$ , so that  $\mathbf{E}_\Lambda = \mathbf{R}_\Lambda^H$  is the corresponding extension matrix. Then, setting

$$\mathbf{A}_\Lambda = \mathbf{R}_\Lambda \mathbf{A} \mathbf{R}_\Lambda^H, \quad (6.8)$$

we preliminary observe that problem (3.4) can be equivalently written as

$$\mathbf{A}_\Lambda \mathbf{u}_\Lambda = \mathbf{f}_\Lambda. \quad (6.9)$$

Next, let  $\mathbf{P}_\Lambda : \ell^2(\mathbb{N}_2) \rightarrow \ell^2(\mathbb{N}_2)$  be the projector operator defined as

$$(\mathbf{P}_\Lambda \mathbf{v})_\lambda = \begin{cases} v_\lambda & \text{if } \lambda \in \Lambda, \\ 0 & \text{if } \lambda \notin \Lambda. \end{cases}$$

Note that  $\mathbf{P}_\Lambda$  can be represented as a diagonal semi-infinite matrix whose diagonal elements are 1 for indexes belonging to  $\Lambda$ , and zero otherwise. Let us set  $\mathbf{Q}_\Lambda = \mathbf{I} - \mathbf{P}_\Lambda$  and introduce the semi-infinite matrix  $\hat{\mathbf{A}}_\Lambda := \mathbf{P}_\Lambda \mathbf{A} \mathbf{P}_\Lambda + \mathbf{Q}_\Lambda$  which is equal to  $\mathbf{A}_\Lambda$  for indexes in  $\Lambda$  and to the identity matrix, otherwise. The definitions of the projectors  $\mathbf{P}_\Lambda$  and  $\mathbf{Q}_\Lambda$  easily yield the following result.

**Property 6.1** (invertibility of  $\hat{\mathbf{A}}_\Lambda$ ). *If  $\mathbf{A}$  is invertible with  $\mathbf{A} \in \mathcal{D}_e(\eta_L)$ , then the same holds for  $\hat{\mathbf{A}}_\Lambda$ .*  $\square$

Finally, the infinite dimensional version of the Galerkin problem (3.4) reads as follows: find  $\hat{\mathbf{u}} \in \ell^2(\mathbb{N}_2)$  such that

$$\hat{\mathbf{A}}_\Lambda \hat{\mathbf{u}} = \mathbf{P}_\Lambda \mathbf{f} . \quad (6.10)$$

Let  $\mathbf{E}_\Lambda = \mathbf{R}_\Lambda^H : \mathbb{R}^{|\Lambda|} \rightarrow \ell^2(\mathbb{N}_2)$  be the extension operator and let  $\mathbf{u}_\Lambda \in \mathbb{R}^{|\Lambda|}$  be the Galerkin solution to (6.9); then, it is easy to check that  $\hat{\mathbf{u}} = \mathbf{E}_\Lambda \mathbf{u}_\Lambda$ .

We are now ready to state the main result of this section.

**Proposition 6.2** (sparsity class of the residual). *Let  $\mathbf{A} \in \mathcal{D}_e(\eta_L)$  and  $\mathbf{A}^{-1} \in \mathcal{D}_e(\bar{\eta}_L)$ , for constants  $\eta_L > 0$  and  $\bar{\eta}_L \in (0, \eta_L]$  according to Property 3.5. If  $u \in \mathcal{A}_G^{\eta, t}$  for some  $\eta > 0$  and  $t \in (0, 1]$ , such that  $\eta < \bar{\eta}_L$ , then there exist suitable positive constants  $\bar{\eta} \leq \eta$  and  $\bar{t} \leq t$  such that  $r(u_\Lambda) \in \mathcal{A}_G^{\bar{\eta}, \bar{t}}$  for any index set  $\Lambda$ , with*

$$\|r(u_\Lambda)\|_{\mathcal{A}_G^{\bar{\eta}, \bar{t}}} \lesssim \|u\|_{\mathcal{A}_G^{\eta, t}} .$$

*Proof.* This is an adaptation of a similar proof in the periodic case given in [5, Proposition 5.4]; we report here the details for completeness. Assume for the moment we are given  $\bar{\eta}$  and  $\bar{t}$ . By using Proposition 6.1, i.e.  $\|\mathbf{A}\mathbf{v}\|_{\ell_G^{\bar{\eta}, \bar{t}}} \lesssim \|\mathbf{v}\|_{\ell_G^{\bar{\eta}_1, \bar{t}_1}}$ , and Property 5.1, we get

$$\begin{aligned} \|\mathbf{r}_\Lambda\|_{\ell_G^{\bar{\eta}, \bar{t}}(\mathbb{N}_2)} &= \|\mathbf{A}(\mathbf{u} - \mathbf{u}_\Lambda)\|_{\ell_G^{\bar{\eta}, \bar{t}}(\mathbb{N}_2)} \lesssim \|\mathbf{u} - \mathbf{u}_\Lambda\|_{\ell_G^{\eta_1, t_1}(\mathbb{N}_2)} \\ &\lesssim \|\mathbf{u}\|_{\ell_G^{2^{t_1}\eta_1, t_1}(\mathbb{N}_2)} + \|\mathbf{u}_\Lambda\|_{\ell_G^{2^{t_1}\eta_1, t_1}(\mathbb{N}_2)}, \end{aligned} \quad (6.11)$$

where  $t_1$  and  $\eta_1$  are defined by the relations

$$\bar{t} = \frac{t_1}{1 + t_1} < t_1, \quad \bar{\eta} = \zeta(t_1)\eta_1 .$$

From (6.10) we have  $\mathbf{u}_\Lambda = (\hat{\mathbf{A}}_\Lambda)^{-1}(\mathbf{P}_\Lambda \mathbf{f})$ . Using Property 6.1 and applying Proposition 6.1 to  $(\hat{\mathbf{A}}_\Lambda)^{-1}$  we get

$$\|\mathbf{u}_\Lambda\|_{\ell_G^{2^{t_1}\eta_1, t_1}(\mathbb{N}_2)} = \|\hat{\mathbf{u}}\|_{\ell_G^{2^{t_1}\eta_1, t_1}(\mathbb{N}_2)} = \|(\hat{\mathbf{A}}_\Lambda)^{-1}(\mathbf{P}_\Lambda \mathbf{f})\|_{\ell_G^{2^{t_1}\eta_1, t_1}(\mathbb{N}_2)} \lesssim \|\mathbf{P}_\Lambda \mathbf{f}\|_{\ell_G^{\eta_2, t_2}(\mathbb{N}_2)} \leq \|\mathbf{f}\|_{\ell_G^{\eta_2, t_2}(\mathbb{N}_2)} ,$$

with

$$2^{t_1}\eta_1 = \zeta(t_2)\eta_2 < \eta_2 , \quad t_1 = \frac{t_2}{1 + t_2} < t_2 .$$

By substituting the above inequality into (6.11) and using again Proposition 6.1 we get

$$\begin{aligned} \|\mathbf{r}_\Lambda\|_{\ell_G^{\bar{\eta}, \bar{t}}(\mathbb{N}_2)} &\lesssim \|\mathbf{u}\|_{\ell_G^{2^{t_1}\eta_1, t_1}(\mathbb{N}_2)} + \|\mathbf{f}\|_{\ell_G^{\eta_2, t_2}(\mathbb{N}_2)} \\ &= \|\mathbf{u}\|_{\ell_G^{2^{t_1}\eta_1, t_1}(\mathbb{N}_2)} + \|\mathbf{A}\mathbf{u}\|_{\ell_G^{\eta_2, t_2}(\mathbb{N}_2)} \lesssim \|\mathbf{u}\|_{\ell_G^{\eta, t}(\mathbb{N}_2)} \end{aligned} \quad (6.12)$$

where

$$\eta_2 = \zeta(t)\eta < \eta , \quad t_2 = \frac{t}{1 + t} < t .$$

This shows that the assertion holds true for the choice

$$\bar{\eta} = \left(\frac{1}{2}\right)^{\frac{t}{1+2t}} \zeta\left(\frac{t}{1+2t}\right) \zeta\left(\frac{t}{1+t}\right) \zeta(t)\eta, \quad \bar{t} = \frac{t}{1+3t} .$$

It remains to verify the assumptions of Proposition 6.1 when  $\mathbf{A}$  is dense. We note that there holds

$$t_1 = \frac{t}{1+2t} < t_2 = \frac{t}{1+t} < t.$$

Moreover, using  $\eta_1 < 2^{t_1}\eta_1 < \eta_2 < \eta$  and  $\eta_L \geq \bar{\eta}_L > \eta$  yields

$$\eta < \eta_L, \quad \eta_1 < \eta_L, \quad \eta_2 < \bar{\eta}_L,$$

which are the required conditions to apply Proposition 6.1 when  $\mathbf{A}$  is dense. This concludes the proof.  $\square$

## 7 The predictor-corrector adaptive algorithm

In this section we study a variant (named **PC-ADLEG**) of the ideal algorithm **ADLEG** introduced above. Motivated by the enlightening discussion at the end of Section 4 and the subsequent results of Sections 5 and 6, we devise each iteration of the algorithm as formed by a predictor step followed by a corrector step. The predictor step guarantees an arbitrarily large error reduction (by suitably enriching the output set from the Dörfler procedure). This step is driven by the sparsity class of the residual and so, in view of Proposition 6.2 it does not guarantee optimality with respect to the sparsity class of the exact solution. The corrector step is realized by introducing a coarsening procedure which removes the smallest components of the output of the predictor step, in such a way to guarantee optimality.

### 7.1 Enrichment

We introduce the procedure **ENRICH** defined as follows:

- $\Lambda^* := \mathbf{ENRICH}(\Lambda, J)$

Given an integer  $J \geq 0$  and a finite set  $\Lambda \subset \mathbb{N}_2$ , the output is the set

$$\Lambda^* := \{k \in \mathbb{N}_2 : \text{there exists } \ell \in \Lambda \text{ such that } |k - \ell| \leq J\}.$$

Note that since the procedure adds a 1-dimensional ball of radius  $J$  around each point of  $\Lambda$ , the cardinality of the new set  $\Lambda^*$  can be estimated as

$$|\Lambda^*| \leq 2J|\Lambda|. \quad (7.1)$$

Recall now that  $\psi_{\mathbf{A}}(J, \eta) = C_{\mathbf{A}}e^{-\eta_L J}$  from Property (3.6). Let  $J_\theta > 0$  be defined as

$$J_\theta := \min \left\{ J \in \mathbb{N} : \psi_{\mathbf{A}^{-1}}(J_\theta, \bar{\eta}_L) = C_{\mathbf{A}^{-1}}e^{-\bar{\eta}_L J_\theta} \leq \sqrt{\frac{1-\theta^2}{\alpha_*\alpha^*}} \right\}. \quad (7.2)$$

- $\Lambda^* := \mathbf{E-DÖRFLER}(r, \theta)$

Given  $\theta \in (0, 1)$  and an element  $r \in H^{-1}(I)$ , the output  $\Lambda^* \subset \mathbb{N}_2$  is defined by the sequence

$$\begin{aligned} \tilde{\Lambda} &:= \mathbf{DÖRFLER}(r, \theta) \\ \Lambda^* &:= \mathbf{ENRICH}(\tilde{\Lambda}, J_\theta). \end{aligned} \quad (7.3)$$

## 7.2 Coarsening

We introduce the new procedure **COARSE** defined as follows:

- $\Lambda := \mathbf{COARSE}(w, \epsilon)$

Given a function  $w \in V_{\Lambda^*}$  for some finite index set  $\Lambda^*$ , and an accuracy  $\epsilon > 0$  which is known to satisfy  $\|u - w\| \leq \epsilon$ , the output  $\Lambda \subseteq \Lambda^*$  is a set of minimal cardinality such that

$$\|w - P_{\Lambda}w\| \leq 2\epsilon. \quad (7.4)$$

The following result shows that the cardinality  $|\Lambda|$  is optimal relative to the sparsity class of  $u$ . We refer to Cohen [9, Theorem 4.9.1] and Stevenson [20, Proposition 3.2].

**Theorem 7.1** (cardinality after coarsening). *Let  $\varepsilon > 0$  and let  $u \in \mathcal{A}_G^{\eta, t}$ . Then*

$$|\Lambda| \leq \frac{1}{\eta^{1/t}} \left( \log \frac{\|u\|_{\mathcal{A}_G^{\eta, t}}}{\varepsilon} \right)^{1/t} + 1.$$

The approximation error obtained after a call of **COARSE** is estimated as follows.

**Property 7.1** (error after coarsening). *The procedure **COARSE** guarantees the bounds*

$$\|u - P_{\Lambda}w\| \leq 3\epsilon \quad (7.5)$$

and, for the Galerkin solution  $u_{\Lambda} \in V_{\Lambda}$ ,

$$\|u - u_{\Lambda}\| \leq 3\sqrt{\alpha^*}\epsilon. \quad (7.6)$$

*Proof.* The first bound is trivial, the second one follows from the minimality property of the Galerkin solution in the energy norm and from (3.3):

$$\|u - u_{\Lambda}\| \leq \|u - P_{\Lambda}w\| \leq \sqrt{\alpha^*}\|u - P_{\Lambda}w\| \leq 3\sqrt{\alpha^*}\epsilon. \quad \square$$

## 7.3 PC-ADLEG: a predictor-corrector version of ADLEG

Given two parameters  $\theta \in (0, 1)$  and  $tol \in [0, 1]$ , we define the following adaptive algorithm.

**Algorithm PC-ADLEG**( $\theta, tol$ )

Set  $r_0 := f$ ,  $\Lambda_0 := \emptyset$ ,  $n = -1$

do

$n \leftarrow n + 1$

$\widehat{\partial\Lambda}_n := \mathbf{E-DÖRFLER}(r_n, \theta)$

$\widehat{\Lambda}_{n+1} := \Lambda_n \cup \widehat{\partial\Lambda}_n$

$\widehat{u}_{n+1} := \mathbf{GAL}(\widehat{\Lambda}_n)$

$\Lambda_{n+1} := \mathbf{COARSE}\left(\widehat{u}_{n+1}, \frac{2}{\alpha_*}\sqrt{1-\theta^2}\|r_n\|\right)$

$u_{n+1} := \mathbf{GAL}(\Lambda_{n+1})$

$$r_{n+1} := \mathbf{RES}(u_{n+1})$$

while  $\|r_{n+1}\| > \text{tol}$

**Theorem 7.2** (contraction property of **PC-ADLEG**). *Let  $0 < \theta < 1$  be chosen so that*

$$\rho = \rho(\theta) = 6 \frac{\alpha^*}{\alpha_*} \sqrt{1 - \theta^2} < 1. \quad (7.7)$$

*If the assumptions of Property 3.5 are fulfilled, the sequence of errors  $u - u_n$  generated for  $n \geq 0$  by the algorithm satisfies the inequality*

$$\|u - u_{n+1}\| \leq \rho \|u - u_n\|.$$

*Thus, for any  $\text{tol} > 0$  the algorithm terminates in a finite number of iterations, whereas for  $\text{tol} = 0$  the sequence  $u_n$  converges to  $u$  in  $H^1(I)$  as  $n \rightarrow \infty$ .*

*Proof.* At the  $n$ -th step, we have  $\widehat{\partial\Lambda}_n = \mathbf{E-DÖRFLER}(r_n\theta)$  and  $\widehat{\Lambda}_{n+1} = \Lambda_n \cup \partial\Lambda_n$ , where

$$\begin{aligned} \widetilde{\partial\Lambda}_n &= \mathbf{DÖRFLER}(r_n, \theta) \\ \widehat{\partial\Lambda}_n &= \mathbf{ENRICH}(\widetilde{\partial\Lambda}_n, J_\theta). \end{aligned} \quad (7.8)$$

We recall that the set  $\widetilde{\partial\Lambda}_n$  is such that  $g_n = P_{\widetilde{\partial\Lambda}_n} r_n$  satisfies

$$\|r_n - g_n\| \leq \sqrt{1 - \theta^2} \|r_n\|$$

(see (4.4)). Let  $w_n \in V$  be the solution of  $Lw_n = g_n$ , which in general will have infinitely many components, and let us split it as

$$w_n = P_{\widehat{\Lambda}_{n+1}} w_n + P_{\widehat{\Lambda}_{n+1}^c} w_n =: y_n + z_n \in V_{\widehat{\Lambda}_{n+1}} \oplus V_{\widehat{\Lambda}_{n+1}^c}.$$

Then, by the minimality property of the Galerkin solution  $\widehat{u}_{n+1} \in V_{\widehat{\Lambda}_{n+1}}$  in the energy norm, and by (3.3) and (3.7), one has

$$\begin{aligned} \|u - \widehat{u}_{n+1}\| &\leq \|u - (u_n + y_n)\| \leq \|u - u_n - w_n + z_n\| \\ &\leq \frac{1}{\sqrt{\alpha_*}} \|L(u - u_n - w_n)\| + \sqrt{\alpha^*} \|z_n\| = \frac{1}{\sqrt{\alpha_*}} \|r_n - g_n\| + \sqrt{\alpha^*} \|z_n\|. \end{aligned}$$

Thus,

$$\|u - \widehat{u}_{n+1}\| \leq \frac{1}{\sqrt{\alpha_*}} \sqrt{(1 - \theta^2)} \|r_n\| + \sqrt{\alpha^*} \|z_n\|.$$

Since  $z_n = (P_{\widehat{\Lambda}_{n+1}^c} L^{-1} P_{\widetilde{\partial\Lambda}_n}) r_n$ , observing that

$$k \in \widehat{\Lambda}_{n+1}^c \quad \text{and} \quad \ell \in \widetilde{\partial\Lambda}_n \quad \Rightarrow \quad |k - \ell| > J_\theta,$$

we have

$$\|P_{\widehat{\Lambda}_{n+1}^c} L^{-1} P_{\widetilde{\partial\Lambda}_n}\| \leq \|\mathbf{A}^{-1} - (\mathbf{A}^{-1})_{J_\theta}\| \leq \psi_{\mathbf{A}^{-1}}(J_\theta, \bar{\eta}_L) \leq \sqrt{\frac{1 - \theta^2}{\alpha_* \alpha^*}},$$

where we have used (7.2). Thus, we obtain

$$\|u - \hat{u}_{n+1}\| \leq \frac{2}{\sqrt{\alpha_*}} \sqrt{1 - \theta^2} \|r_n\| \quad (7.9)$$

or, equivalently,

$$\|u - \hat{u}_{n+1}\| \leq \frac{2}{\alpha_*} \sqrt{1 - \theta^2} \|r_n\|. \quad (7.10)$$

Since the right-hand side of (7.10) is precisely the parameter  $\epsilon_n$  fed to the procedure **COARSE**, Property (7.1) implies

$$\|u - u_{n+1}\| \leq 6 \frac{\sqrt{\alpha_*}}{\alpha_*} \sqrt{1 - \theta^2} \|r_n\| \leq 6 \frac{\alpha_*}{\alpha_*} \sqrt{1 - \theta^2} \|u - u_n\|, \quad (7.11)$$

for the Galerkin solution  $u_{n+1} \in V_{\Lambda_{n+1}}$ . The assertion thus follows immediately.  $\square$

The rest of the paper will be devoted to investigating complexity issues for the sequence of approximations  $u_n = u_{\Lambda_n}$  generated by **PC-ADLEG**. In particular, we wish to estimate the cardinality of each  $\Lambda_n$  and check whether its growth is “optimal” with respect to the sparsity class  $\mathcal{A}_\phi$  of the exact solution, in the sense that  $|\Lambda_n|$  is comparable to the cardinality of the index set of the best approximation of  $u$  yielding the same error  $\|u - u_n\|$ .

**Theorem 7.3** (cardinality of **PC-ADLEG**). *Suppose that  $u \in \mathcal{A}_G^{\eta,t}$ , for some  $\eta > 0$  and  $t \in (0, 1]$ . Then, there exists a constant  $C > 1$  such that the cardinality of the set  $\Lambda_n$  of the active degrees of freedom produced by **PC-ADLEG** satisfies the bound*

$$|\Lambda_n| \leq \frac{1}{\eta^{1/t}} \left( \log \frac{\|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u - u_n\|} + \log C \right)^{1/t} + 1, \quad \forall n \geq 0.$$

If, in addition, the assumptions of Proposition 6.2 are satisfied, then the cardinality of the intermediate sets  $\hat{\Lambda}_{n+1}$  activated in the predictor step can be estimated as

$$|\hat{\Lambda}_{n+1}| \leq |\Lambda_n| + \frac{2J_\theta}{\bar{\eta}^{1/\bar{t}}} \left( \log \frac{\|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u - u_{n+1}\|} + \log C \right)^{1/\bar{t}} + 2J_\theta, \quad \forall n \geq 0,$$

where  $J_\theta$  is defined in (7.2) and  $\bar{\eta} \leq \eta$ ,  $\bar{t} \leq t$  are the parameters which occur in the thesis of Proposition 6.2.

*Proof.* Combining Theorem 7.1 with  $\epsilon_n = \frac{2}{\alpha_*} \sqrt{1 - \theta^2} \|r_n\|$  and Property 7.1, we get

$$|\Lambda_{n+1}| \leq \frac{1}{\eta^{1/t}} \left( \log \frac{\|u\|_{\mathcal{A}_G^{\eta,t}}}{\epsilon_n} \right)^{1/t} + 1 \leq \frac{1}{\eta^{1/t}} \left( \log \frac{\|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u - u_{n+1}\|} + \log C \right)^{1/t} + 1.$$

We now estimate  $|\hat{\Lambda}_{n+1}|$ . Recalling (7.1), we have  $|\partial \hat{\Lambda}_n| \leq 2J_\theta |\partial \widetilde{\Lambda}_n|$ , and using (6.2) together with Proposition 6.1 and (7.11) yields

$$\begin{aligned} |\partial \widetilde{\Lambda}_n| &\leq \frac{1}{\bar{\eta}^{1/\bar{t}}} \left( \log \frac{\|r_n\|_{\mathcal{A}_G^{\bar{\eta},t}}}{\sqrt{1 - \theta^2} \|r_n\|} \right)^{1/\bar{t}} + 1 \\ &\leq \frac{1}{\bar{\eta}^{1/\bar{t}}} \left( \log \frac{\|u\|_{\mathcal{A}_G^{\eta,t}}}{\|u - u_{n+1}\|} + \log C \right)^{1/\bar{t}} + 1. \end{aligned}$$



Finally, the second assertion follows from  $|\widehat{\Lambda}_{n+1}| \leq |\Lambda_n| + |\partial\Lambda_n|$ .  $\square$

We observe that the cardinality of  $\Lambda_n$ , i.e., the set of degrees of freedom of the Galerkin solution at the end of each iteration, is optimal. On the other hand, in the case  $\bar{\eta} < \eta$  and  $\bar{t} < t$ , the cardinality of  $\widehat{\Lambda}_{n+1}$  may grow at a faster rate than the cardinality of  $\Lambda_n$ . This is a direct consequence of the fact that, according to Proposition 6.2, the sparsity class of the residual may be worse than that of the solution. Moreover, the contraction constant in (7.7) is as close to 0 as desired provided  $\theta$  is close to 1, which is consistent with the expected fast error decay of spectral methods. This is a key feature of our contribution and a novel idea with respect to the standard algebraic case; see the surveys [16, 20].

Similar results to Theorems 7.2 and 7.3 are valid also for the algebraic case, with optimal convergence rates only limited by solution regularity; they can be derived as in [5, Theorems 3.3 and 7.2]. In addition, a more conservative version of **PC-ADLEG**, which avoids the procedure **ENRICH**, can be studied as well following [5, Theorem 8.1]. This version exhibits better optimality properties than **PC-ADLEG** at the expense of a relatively large contraction factor, which is at odds with spectral accuracy.

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